International Journal of Solids and Structures 50 (2013) 396–402

Contents lists available at SciVerse ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Regulating the dynamic behavior of a column with uncertain initial imperfections by support-placing

Isaac Elishakoff [a], Xiaojun Wang [b,*], Yunlong Li [b], Juxi Hu [b], Zhiping Qiu [b]

[a] Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA
[b] Institute of Solid Mechanics, Beihang University, Beijing 100191, PR China

ARTICLE INFO

Article history:
Received 7 January 2012
Received in revised form 20 August 2012
Available online 23 October 2012

Keywords:
Two-span column
Dynamic response
Anti-optimization
Uncertain imperfection
Convex models

ABSTRACT

Dynamic behavior of two-span columns with uncertain imperfections is studied. The initial imperfections are modeled as belonging to ellipsoidal set. The maximum and minimum responses of the structure are determined via the convex analysis. The maximum response then is regulated by suitable choice of the support location. Specifically, we determine the location of the support so as to yield the minimum of the maximum response. By this means we minimize, by support placing, the worst, anti-optimized response of the structure, and thus increase its safety. Whereas there are studies devoted to uncertain imperfections, as well as to free vibration of structures with additional supports, the present study appears to be the first one that introduces the idea of regulation by support placing in structures with uncertain imperfections.

1. Introduction

During exploitation of various structures the problem of regulating their dynamic behavior often arises. Namely, researchers study variation patterns of different response parameters of the structure so that it exhibits the maximum, the minimum, or a preselected behavior. This task is most frequently performed via structural optimization, i.e., choosing the suitably varying cross-sectional area or mechanical characteristics so that the desired response or eigenvalue is achieved. Regulation via proper assigning of the cross-sectional area was studied, for example, by Katsikadelis and Tsiatas (2006). The choice of elastic modulus variation so that the tailored, desired behavior is resulting was studied by Elishakoff (2005) in his monograph, where various problems of vibration and buckling were solved so as to achieve the eigenvalue that is either below or above a specified level.

There is an alternative way of regulating the vibration behavior. It is achieved by placing additional support(s) at suitable location(s). This does not necessarily demand the shape optimization to be conducted in conjunction with it. Such a regulation was studied in static and in buckling contexts, in the book by Abovskii et al. (1993), and well as in several papers namely, by Prager and Rozvany (1975), Rozvany and Mroz (1977), Szelag and Uróz (1978), Åkesson and Olhoff (1988), Dekhtiar (1991), Wang et al. (1992), Liu et al. (1994), Won and Park (1998), Neuringer and Elishakoff (1998), Liu, Hu and Huang (2000) and others.

In this study, we regulate the behavior by changing the location of the additional support, by this means we minimize the least favorable response. We investigate a two-span beam that possesses initial imperfections. The initial imperfections constitute an uncertain field such that the corresponding Fourier coefficients belong to the ellipsoidal set. We determine maximum and minimum possible displacement of the beam subject to the constraint that the Fourier coefficients of the imperfections vary within a given ellipsoid but otherwise are arbitrary. These responses depend explicitly on the support location. From the point of view of structural safety the maximum structural response is the characteristic that is responsible for possible failure. Hence it should be made as small as possible. The purpose of this study is to find the optimum location of the support. The general methodology of this paper falls within combined anti-optimization (Ben-Haim and Elishakoff, 1990; Qiu and Wang, 2005) and optimization (Elishakoff and Ohsaki, 2010). Here the additional support location is chosen as the optimizing, regulating parameter of the anti-optimized response. It must be stressed that to the best of the author's knowledge this is the first study that deals with regulation of dynamic behavior of imperfection structures.

2. Free vibration of perfect two-span column

We first consider the free vibration of an uniform beam of length L with an additional support placed at the location $x = a$. The governing differential equation reads

* Corresponding author. Fax: +86 10 82313658.
E-mail address: xjwang@buaa.edu.cn (X. Wang).

0020-7683/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijsolstr.2012.10.003

$$EI\frac{d^4w}{dx^4} - \rho A\omega^2 w = 0 \tag{1}$$

where $w(x)$ = mode shape, E = modulus of elasticity, I = moment of inertia, ρ = material density, A = cross-sectional area, ω = natural frequency, x = axial coordinate.

The solution for the mode shape is denoted as $w_1(x)$ in the region $0 \leqslant x \leqslant a$, whereas it is designated as $w_2(x)$ in the region $a \leqslant x \leqslant L$. We introduce two local coordinate systems. The first system is associated with the first span of the beam. The axial coordinate in the first span is denoted x_1; it varies in the range $0 \leqslant x_1 \leqslant a$. The axial coordinate in the second span is denoted x_2. It varies in the range $0 \leqslant x_2 \leqslant L - a$. The boundary conditions read:

$$\begin{aligned} &w_1 = 0, \quad \tfrac{d^2w_1}{dx^2} = 0 \quad at \quad x_1 = 0 \\ &w_2 = 0, \quad \tfrac{d^2w_2}{dx^2} = 0 \quad at \quad x_2 = L - a \end{aligned} \tag{2}$$

The continuity conditions are

$$\begin{aligned} &w_1(a) = w_2(0) = 0 \\ &\frac{dw_1(a)}{dx_1} = \frac{dw_2(0)}{dx_2} \\ &\frac{d^2w_1(a)}{dx_1^2} = \frac{d^2w_2(0)}{dx_2^2} \end{aligned} \tag{3}$$

The expressions for $w_1(x_1)$ and $w_2(x_2)$ are

$$w_1(x_1) = C_1\sin(kx_1) + C_2\cos(kx_1) + C_3\sinh(kx_1) + C_4\cosh(kx_1) \tag{4}$$

$$w_2(x_2) = D_1\sin(kx_2) + D_2\cos(kx_2) + D_3\sinh(kx_2) + D_4\cosh(kx_2) \tag{5}$$

where C_j and D_j are constants of integration, and

$$k = \sqrt[4]{\rho A\omega^2/EI} \tag{6}$$

Satisfaction of the four boundary conditions and four continuity conditions for eight constants of integration leads to the transcendental characteristic equation

$$-\sin(kL)\sinh(ka)\sinh[k(L-a)] + \sin(ka)\sin[k(L-a)]\sinh(kL) = 0 \tag{7}$$

With notation $kL = \gamma$, we get a transcendental equation which contains the ratio $\alpha = a/L$, of the additional support location to the total length of the beam, as a parameter:

$$-\sin(\gamma)\sinh(\gamma\alpha)\sinh[\gamma(1-\alpha)] + \sin(\gamma\alpha)\sin[\gamma(1-\alpha)]\sinh(\gamma) = 0 \tag{8}$$

The corresponding mode shapes read

$$W_{1n}(\eta_1) = \bar{C}_1\sin(\gamma_n\eta_1) + \bar{C}_3\sinh(\gamma_n\eta_1), \quad \text{for } 0 \leqslant \eta_1 \leqslant \alpha \tag{9}$$

$$\begin{aligned} W_{2n}(\eta_2) = {} & \bar{D}_1\sin(\gamma_n\eta_2) + \bar{D}_2[\cos(\gamma_n\eta_2) - \cosh(\gamma_n\eta_2)] \\ & + \bar{D}_3\sinh(\gamma_n\eta_2), \text{for } 0 \leqslant \eta_2 \leqslant 1 - \alpha \end{aligned} \tag{10}$$

where $\eta_i = x_i/L$ is the non-dimensional axial coordinate in ith coordinate system. Moreover,

$$\begin{aligned} &\bar{C}_1 == 1 \\ &\bar{C}_3 = -\tanh[\gamma(1-\alpha)]/\tanh[\gamma(1-\alpha)] \\ &\bar{D}_1 = -[1 + \tanh(\gamma(1-\alpha))\cdot(\coth(\gamma\alpha) - \cot(\gamma\alpha))]\sin(\gamma\alpha)/\tanh[\gamma(1-\alpha)] \\ &\bar{D}_2 = \tanh[\gamma(1-\alpha)]\sin(\gamma\alpha)/\tanh[\gamma(1-\alpha)] \\ &\bar{D}_3 = \sin(\gamma\alpha)/\tanh[\gamma(1-\alpha)] \end{aligned} \tag{11}$$

3. Vibration of a column with initial imperfection

Consider now the case when the column is not perfect; in other words, it contains geometric imperfections.

The governing differential equation in new circumstance reads

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = -P\frac{d^2 w_0}{dx^2} \tag{12}$$

where $w_0(x)$ is an initial imperfection, namely, the deviation of the column's axis from the straight position, P is the axial force, t = time. In addition to the boundary conditions in Eq. (2) and compatibility conditions in Eq. (3) one also imposes initial conditions, as follows:

$$w(x,t) = \frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{13}$$

We introduce following non-dimensional quantities

$$\phi = w_0(x)/\Delta, \quad u = w(x,t)/\Delta, \quad \eta = x/L, \quad \eta \in [0,1] \tag{14}$$

where Δ = radius of inertia of the column's cross-section $\Delta = (I/A)^{1/2}$; η = non-dimensional axial coordinate; moreover, the non-dimensional time coordinate is chosen as

$$\tau = \omega_1 t \tag{15}$$

where

$$\omega_1 = \left(\frac{\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}} \tag{16}$$

is the fundamental natural frequency of the associated *perfect* beam that is simply supported at its ends $x = 0$ and $x = L$ *without* additional support and *without* axial force. In addition, a non-dimensional force ratio is introduced,

$$\delta = P/P_{cr} \tag{17}$$

where

$$P_{cr} = \pi^2 EI/L^2 \tag{18}$$

is the critical, Euler load of an uniform column of length L that is simply supported at both ends, without an intermediate support.

The non-dimensional governing differential equation becomes

$$\frac{\partial^4 u}{\partial\eta^4} + \pi^2\delta\frac{\partial^2 u}{\partial\eta^2} + \pi^4\frac{\partial^2 u}{\partial\tau^2} = -\pi^2\delta\frac{\partial^2\phi}{\partial\eta^2} \tag{19}$$

We will first deal with the solution of the deterministic problem when the initial imperfection function w_0 in Eq. (12) or ϕ in Eq. (19) are fully specified.

4. Solution of deterministic problem

We expand the initial imperfection function $\phi(\eta)$ in terms of the mode shapes $W_n(\eta)$ of the associated problem without initial imperfections:

$$\phi(\eta) = \sum_{n=1}^{\infty} A_n W_n(\eta) \tag{20}$$

Alternatively, we write for each span

$$\phi(\eta) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} A_n W_{1n}(\eta), & \text{for} \quad 0 \leqslant \eta \leqslant \alpha \\ \displaystyle\sum_{n=1}^{\infty} A_n W_{2n}(\eta), & \text{for} \quad 0 \leqslant \eta \leqslant 1 - \alpha \end{cases} \tag{21}$$

Likewise, the additional displacement $u(\eta,\tau)$ is expanded as follows

$$u(\eta,\tau) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} G_n(\tau) W_{1n}(\eta), & \text{for} \quad 0 \leqslant \eta \leqslant \alpha \\ \displaystyle\sum_{n=1}^{\infty} G_n(\tau) W_{2n}(\eta), & \text{for} \quad 0 \leqslant \eta \leqslant 1 - \alpha \end{cases} \tag{22}$$

For simplicity of further computation, we consider the case in which the initial imperfection constitutes a linear combination of the first two free vibration modes, so that in Eq. (20) instead of infinite number of terms, only two terms are present. Thus, we have to also retain two terms in the series in Eq. (22). We substitute Eqs. (21) and (22) into the governing differential Eq. (19), multiply the result by the function $W_1(\eta)$ and integrate from 0 to 1. We get the following ordinary, second-order differential equation for $G_i(\tau)$ with respect to the non-dimension time coordinate:

$$\frac{d^2G_1(\tau)}{d\tau^2}+[j_1+\delta j_2]G_1(\tau)+\delta j_3G_2(\tau)=-\delta j_2A_1-\delta j_3A_2 \tag{23}$$

Likewise, substituting Eqs. (21) and (22) into Eq. (19) and multiplying the result by $W_2(\eta)$ and integrating over the length of the column yields, a second equation:

$$\frac{d^2G_2(\tau)}{d\tau^2}+[h_1+\delta h_2]G_2(\tau)+\delta h_3G_1(\tau)=-\delta h_3A_1-\delta h_2A_2 \tag{24}$$

In Eqs. (23) and (24) the following notation is utilized for the inner product of two functions $f_1(\eta)$ and $f_2(\eta)$

$$(f_1,f_2)=\int_0^1 f_1(\eta)f_2(\eta)d\eta \tag{25}$$

In addition,

$$\begin{aligned} j_1&=\frac{(W_1^{\mathrm{IV}},W_1)}{\pi^4(W_1,W_1)},\quad j_2=\frac{(W_1'',W_1)}{\pi^2(W_1,W_1)},\quad j_3=\frac{(W_2'',W_1)}{\pi^2(W_1,W_1)}\\ h_1&=\frac{(W_2^{\mathrm{IV}},W_2)}{\pi^4(W_2,W_2)},\quad h_2=\frac{(W_2'',W_2)}{\pi^2(W_2,W_2)},\quad h_3=\frac{(W_1'',W_2)}{\pi^2(W_2,W_2)}\end{aligned} \tag{26}$$

We look for the complementary solution in the form

$$G_1^c(\tau)=Be^{\lambda\tau},\quad G_2^c(\tau)=Ce^{\lambda\tau} \tag{27}$$

From Eqs. (23) and (24), we obtain

$$\begin{aligned}&B(\lambda^2+j_1+\delta j_2)+C\delta j_3=0\\ &B\delta h_3+C(\lambda^2+h_1+\delta h_2)=0\end{aligned} \tag{28}$$

leading to the determinantal equation

$$\begin{vmatrix}\lambda^2+j_1+\delta j_2 & \delta j_3\\ \delta h_3 & \lambda^2+h_1+\delta h_2\end{vmatrix}=0 \tag{29}$$

which represents a biquadratic equation for λ

$$\lambda^4+\lambda^2(h_1+\delta h_2+j_1+\delta j_2)+(j_1+\delta j_2)(h_1+\delta h_2)-\delta^2J_3h_3=0 \tag{30}$$

The solution for λ^2 reads

$$\lambda_{\mathrm{I,II}}^2=\frac{1}{2}\left[-h_1-j_1-h_2\delta-j_2\delta\mp\sqrt{R}\right] \tag{31}$$

where

$$\begin{aligned}R&=(h_1+j_1+h_2\delta+j_2\delta)^2\\ &\quad-4(h_1J_1+h_2J_1\delta+h_1J_2\delta+h_2J_2\delta^2-h_3J_3\delta^2)\end{aligned} \tag{32}$$

is the discriminant. We are interested with the roots of the equation

$$R=0 \tag{33}$$

with respect to δ. The roots δ_i ($i=1,2$) can be written as

$$\delta_{1,2}=\frac{-h_1h_2+h_2J_1+h_1J_2-j_1J_2\mp2\sqrt{s}}{h_2^2-2h_2J_2+j_2^2+4h_3J_3} \tag{34}$$

where s is the discriminant

$$s=-h_1^2h_3J_3+2h_1h_3J_1J_3-h_3J_1^2J_3 \tag{35}$$

If $s\leqslant0$, $\delta_{1,2}$ are complex numbers. If $s>0$, we derive two intervals of δ. In one $R<0$, whereas in the other one $R\geqslant0$:

$$R<0\quad\text{for}\quad\delta\in(\delta_1,\delta_2) \tag{36}$$

$$R\geqslant0\quad\text{for}\quad\delta\in[0,\delta_1]\cup[\delta_2,+\infty] \tag{37}$$

From Eq. (31) we deduce that the roots of $\lambda_{\mathrm{I,II}}^2$ in Eq. (31) constitute complex numbers if satisfies condition Eq. (36), and can be deleted (because of Eq. (31)). Now we need to discuss the solution under the condition given in Eq. (37). Let us consider the following equations:

$$\lambda_{\mathrm{I}}^2=\frac{1}{2}\left[-h_1-j_1-h_2\delta-j_2\delta-\sqrt{R}\right]=0 \tag{38}$$

$$\lambda_{\mathrm{II}}^2=\frac{1}{2}\left[-h_1-j_1-h_2\delta-j_2\delta+\sqrt{R}\right]=0 \tag{39}$$

The solution δ of Eq. (38) reads

$$\delta=k_1 \tag{40}$$

whereas the solution δ of Eq. (39) is

$$\delta=k_2 \tag{41}$$

Comparison of these two solutions leads to two possible conclusions:

$$k_1<k_2\quad\text{or}\quad k_1>k_2 \tag{42}$$

From Eq. (28) we get

$$C=-B\frac{\lambda^2+j_1+\delta j_2}{\delta j_3} \tag{43}$$

We consider cases indicated in Eq. (42).

Case 1: $\delta\in(0,k_1),k_1<k_2$ or $\delta\in(0,k_2),k_1>k_2$, the solutions for λ_{I}, λ_{II}, G_1^c and G_2^c are

$$\lambda_{\mathrm{I}}=\pm ir_1=\pm i\frac{\sqrt{\left|h_1+j_1+h_2\delta+j_2\delta+\sqrt{R}\right|}}{\sqrt{2}} \tag{44}$$

$$\lambda_{\mathrm{II}}=\pm ir_2=\pm i\frac{\sqrt{\left|h_1+j_1+h_2\delta+j_2\delta-\sqrt{R}\right|}}{\sqrt{2}} \tag{45}$$

$$G_1^C=B_1\sin(|r_1|\tau)+B_2\cos(|r_1|\tau)+B_3\sin(|r_2|\tau)+B_4\cos(|r_2|\tau) \tag{46}$$

$$\begin{aligned}G_2^C=&-\frac{-r_1^2+j_1+\delta j_2}{\delta j_3}[B_1\sin(|r_1|\tau)+B_2\cos(|r_1|\tau)]\\ &-\frac{-r_2^2+j_1+\delta j_2}{\delta j_3}[B_3\sin(|r_2|\tau)+B_4\cos(|r_2|\tau)]\end{aligned} \tag{47}$$

Case 2: This case is associated with two subcases.

(1) $\delta\in[k_1,k_2];k_1<k_2$, the solutions for λ_{I}, λ_{II}, G_1^c and G_2^c are

$$\lambda_{\mathrm{I}}=\pm r_1=\pm\frac{\sqrt{-h_1-j_1-h_2\delta-j_2\delta-\sqrt{R}}}{\sqrt{2}} \tag{48}$$

$$\lambda_{\mathrm{II}}=\pm i\cdot r_2=\pm i\frac{\sqrt{-h_1-j_1-h_2\delta-j_2\delta+\sqrt{R}}}{\sqrt{2}} \tag{49}$$

$$G_1^c=B_1\sin(r_1\tau)+B_2\cos(r_1\tau)+B_3\sinh(|r_2|\tau)+B_4\cosh(|r_2|\tau) \tag{50}$$

$$G_2^c = -\frac{r_1^2 + j_1 + \delta j_2}{\delta j_3}[B_1 \sin(r_1\tau) + B_2 \cos(r_1\tau)] - \frac{-r_2^2 + j_1 + \delta j_2}{\delta j_3}[B_3 \sinh(r_2\tau) + B_4 \cosh(r_2\tau)] \quad (51)$$

(2) $\delta \in [k_2, k_1], k_1 > k_2$, the solutions for λ_{I}, λ_{II}, G_1^c and G_2^c are

$$\lambda_{\mathrm{I}} = \pm i \cdot r_1 = \pm i \frac{\sqrt{-h_1 - j_1 - h_2\delta - j_2\delta - \sqrt{R}}}{\sqrt{2}} \quad (52)$$

$$\lambda_{\mathrm{II}} = \pm r_2 = \pm \frac{\sqrt{-h_1 - j_1 - h_2\delta - j_2\delta + \sqrt{R}}}{\sqrt{2}} \quad (53)$$

$$G_1^c = B_1 \sin(|r_1|\tau) + B_2 \cos(|r_1|\tau) + B_3 \sinh(r_2\tau) + B_4 \cosh(r_2\tau) \quad (54)$$

$$G_2^c = -\frac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}[B_1 \sin(r_1\tau) + B_2 \cos(r_1\tau)] - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3}[B_3 \sinh(r_2\tau) + B_4 \cosh(r_2\tau)] \quad (55)$$

Case 3: $\delta \in [k_2, +\infty], k_1 < k_2$ or $\delta \in [k_1, +\infty], k_1 > k_2$, the solutions for λ_{I}, λ_{II}, G_1^c and G_2^c are

$$\lambda_{\mathrm{I}} = \pm r_1 = \pm \frac{\sqrt{-h_1 - j_1 - h_2\delta - j_2\delta - \sqrt{R}}}{\sqrt{2}} \quad (56)$$

$$\lambda_{\mathrm{II}} = \pm r_2 = \pm \frac{\sqrt{-h_1 - j_1 - h_2\delta - j_2\delta + \sqrt{R}}}{\sqrt{2}} \quad (57)$$

$$G_1^c = B_1 \sinh(r_1\tau) + B_2 \cosh(r_1\tau) + B_3 \sinh(r_2\tau) + B_4 \cosh(r_2\tau) \quad (58)$$

$$G_2^c = -\frac{r_1^2 + j_1 + \delta j_2}{\delta j_3}[B_1 \sin(r_1\tau) + B_2 \cos(r_1\tau)] - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3} \times [B_3 \sinh(r_2\tau) + B_4 \cosh(r_2\tau)] \quad (59)$$

Now we look for the particular solution of Eqs. (23) and (24) in the form

$$\begin{aligned} G_1^p &= \theta_1 A_1 + \theta_2 A_2 \\ G_2^p &= \theta_3 A_1 + \theta_4 A_2 \end{aligned} \quad (60)$$

We substitute Eq. (60) into Eqs. (23) and (24) to obtain

$$\begin{aligned} &(\theta_1 A_1 + \theta_2 A_2)(j_1 + \delta j_2) + (\theta_3 A_1 + \theta_4 A_2) \cdot \delta j_3 = -\delta j_2 A_1 - \delta j_3 A_2 \\ &(\theta_3 A_1 + \theta_4 A_2)(h_1 + \delta h_2) + (\theta_1 A_1 + \theta_2 A_2) \cdot \delta h_3 = -\delta h_3 A_1 - \delta h_2 A_2 \end{aligned} \quad (61)$$

We derive solution θ_1 and θ_3 of Eq. (61) as follows

$$\theta_1 = \frac{\delta j_2(h_1 + \delta h_2) - \delta^2 J_3 h_3}{\delta^2 J_3 h_3 - (j_1 + \delta j_2)(h_1 + \delta h_2)} \quad (62)$$

$$\theta_2 = \frac{\delta j_3(h_1 + \delta h_2) - \delta^2 J_3 h_2}{\delta^2 J_3 h_3 - (j_1 + \delta j_2)(h_1 + \delta h_2)} \quad (63)$$

$$\theta_3 = \frac{\delta h_3(j_1 + \delta j_2) - \delta^2 J_2 h_3}{\delta^2 J_3 h_3 - (j_1 + \delta j_2)(h_1 + \delta h_2)} \quad (64)$$

$$\theta_4 = \frac{\delta h_2(j_1 + \delta j_2) - \delta^2 J_3 h_3}{\delta^2 J_3 h_3 - (j_1 + \delta j_2)(h_1 + \delta h_2)} \quad (65)$$

The general solution of the inhomogeneous Eqs. (23) and (24) is obtained by the summation of the complementary solution and particular solution:

$$\begin{aligned} G_1(\tau) &= G_1^c(\tau) + G_1^p \\ G_2(\tau) &= G_2^c(\tau) + G_2^p \end{aligned} \quad (66)$$

Under the conditions that imposed at $\tau = 0$, the displacement and its first derivative vanish, thus

$$u(\eta, \tau)|_{\tau=0} = G_1(0)W_{\mathrm{I}}(\eta) + G_2(0)W_{\mathrm{II}}(\eta) = 0 \quad (67)$$

$$\left.\frac{\partial u(\eta, \tau)}{\partial \tau}\right|_{\tau=0} = G_1'(0)W_{\mathrm{I}}(\eta) + G_2'(0)W_{\mathrm{II}}(\eta) = 0 \quad (68)$$

We multiply each side of Eq. (67) with $W_{\mathrm{I}}(\eta)$ and $W_{\mathrm{II}}(\eta)$, respectively, and integrate them along the beam's length, to get

$$G_1(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{I}}(\eta)d\eta + G_2(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{II}}(\eta)d\eta = 0 \quad (69)$$

$$G_1(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{II}}(\eta)d\eta + G_2(0)\int_0^1 W_{\mathrm{II}}(\eta)W_{\mathrm{II}}(\eta)d\eta = 0 \quad (70)$$

Likewise, we multiply each side of Eq. (68) by $W_{\mathrm{I}}(\eta)$ and $W_{\mathrm{II}}(\eta)$, respectively, and integrate them along the beam's length, to obtain

$$G_1'(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{I}}(\eta)d\eta + G_2'(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{II}}(\eta)d\eta = 0 \quad (71)$$

$$G_1'(0)\int_0^1 W_{\mathrm{I}}(\eta)W_{\mathrm{II}}(\eta)d\eta + G_2'(0)\int_0^1 W_{\mathrm{II}}(\eta)W_{\mathrm{II}}(\eta)d\eta = 0 \quad (72)$$

Using Eqs. (69)–(72), we derive initial conditions for $G_1(\tau)$ and $G_2(\tau)$ as follows

$$G_1(0) = G_2(0) = G_1'(0) = G_2'(0) = 0 \quad (73)$$

Case 1: $\delta \in (0, k_1)$, $k_1 < k_2$ or $\delta \in (0, k_2)$, $k_1 > k_2$

$$\begin{aligned} G_1(0) &= B_2 + B_4 + G_1^p = 0 \\ G_2(0) &= -\tfrac{-r_1^2 + j_1 + \delta j_2}{\delta j_3} B_2 - \tfrac{-r_2^2 + j_1 + \delta j_2}{\delta j_3} B_4 + G_2^p = 0 \\ G_1'(0) &= |r_1|B_1 + |r_2|B_3 = 0 \\ G_2'(0) &= \tfrac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}|r_1|B_1 + \tfrac{-r_2^2 + j_1 + \delta j_2}{\delta j_3}|r_2|B_3 = 0 \end{aligned} \quad (74)$$

Using Eq. (74), and letting

$$B_1 = B_3 = 0 \quad (75)$$

we obtain B_2 and B_4 as follow:

$$\begin{aligned} B_2 &= \Re_1 A_1 + \Re_2 A_2 \\ B_4 &= \Re_3 A_1 + \Re_4 A_2 \end{aligned} \quad (76)$$

where

$$\Re_1 = \frac{j_2\delta[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]] - \delta(h_1 J_2 + h_2 J_2\delta - h_3 J_3\delta)r_2^2}{[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]](r_1^2 - r_2^2)}$$

$$\Re_2 = \frac{j_3\delta[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]] - \delta h_1 J_3 r_2^2}{[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]](r_1^2 - r_2^2)}$$

$$\Re_3 = \frac{-j_2\delta[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]] + \delta(h_1 J_2 + h_2 J_2\delta - h_3 J_3\delta)r_1^2}{[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]](r_1^2 - r_2^2)}$$

$$\Re_4 = \frac{-j_3\delta[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]] + \delta h_1 J_3 r_1^2}{[h_1(j_1 + j_2\delta) + \delta[-h_3 J_3\delta + h_2(j_1 + j_2\delta)]](r_1^2 - r_2^2)}$$

Substituting Eqs. (75) and (76) into Eq. (66), and utilizing Eqs. (58) and (59), we determine $G_1(\tau)$ and $G_2(\tau)$ as follow

$$\begin{aligned} G_1(\tau) &= A_1\psi_1(\tau) + A_2\psi_2(\tau) \\ G_2(\tau) &= A_1A_1\psi_3(\tau) + A_2\psi_4(\tau) \end{aligned} \tag{77}$$

where

$$\psi_1(\tau) = \Re_1\cos(|r_1|\tau) + \Re_3\cos(|r_2|\tau) + \theta_1$$

$$\psi_2(\tau) = \Re_2\cos(|r_1|\tau) + \Re_4\cos(|r_2|\tau) + \theta_2$$

$$\psi_3(\tau) = -\frac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}\quad \Re_1\cos(|r_1|\tau) - \frac{-r_2^2 + j_1 + \delta j_2}{\delta j_3} \Re_3\cos(|r_2|\tau) + \theta_3$$

$$\psi_4(\tau) = -\frac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}\quad \Re_2\cos(|r_1|\tau) - \frac{-r_2^2 + j_1 + \delta j_2}{\delta j_3} \Re_4\cos(|r_2|\tau) + \theta_4$$

Case 2: $\delta \in [k_1, k_2], k_1 < k_2$ or $\delta \in [k_2, k_1], k_1 > k_2$

$$\begin{aligned} G_1(0) &= B_2 + B_4 + G_1^{(p)} = 0 \\ G_2(0) &= -\tfrac{-r_1^2+j_1+\delta j_2}{\delta j_3}B_2 - \tfrac{r_2^2+j_1+\delta j_2}{\delta j_3}B_4 + G_2^{(p)} = 0 \\ G_1'(0) &= |r_1|B_1 + r_2B_3 = 0 \\ G_2'(0) &= \tfrac{-r_1^2+j_1+\delta j_2}{\delta j_3}|r_1|B_1 + \tfrac{r_2^2+j_1+\delta j_2}{\delta j_3}r_2B_3 = 0 \end{aligned} \tag{78}$$

Using Eqs. (58), (59), (70), and (72) and solving out for $G_1(\tau)$ and $G_2(\tau)$ we obtain

$$\begin{aligned} G_1(\tau) &= A_1\psi_1(\tau) + A_2\psi_2(\tau) \\ G_2(\tau) &= A_1\psi_3(\tau) + A_2\psi_4(\tau) \end{aligned} \tag{79}$$

where, $\Re_1$, $\Re_2$, $\Re_3$ and $\Re_4$ are as same as case 1.

Moreover

$$\psi_1(\tau) = \Re_1\cos(|r_1|\tau) + \Re_3\cosh(r_2\tau) + \theta_1$$

$$\psi_2(\tau) = \Re_2\cos(|r_1|\tau) + \Re_4\cosh(r_2\tau) + \theta_2$$

$$\psi_3(\tau) = -\frac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}\Re_1\cos(|r_1|\tau) - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3} \Re_3\cosh(r_2\tau) + \theta_3$$

$$\psi_4 = -\frac{-r_1^2 + j_1 + \delta j_2}{\delta j_3}\Re_2\cos(|r_1|\tau) - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3}\quad \Re_4\cosh(r_2\tau) + \theta_4$$

Case 3: $\delta \in [k_2, +\infty], k_1 < k_2$ or $\delta \in [k_1, +\infty], k_1 > k_2$

$$\begin{aligned} G_1(0) &= B_2 + B_4 + G_1^{(p)} = 0 \\ G_2(0) &= -\tfrac{r_1^2+j_1+\delta j_2}{\delta j_3}B_2 - \tfrac{r_2^2+j_1+\delta j_2}{\delta j_3}B_4 + G_1^{(p)} = 0 \\ G_1'(0) &= r_1B_1 + r_2B_3 = 0 \\ G_2'(0) &= \tfrac{r_1^2+j_1+\delta j_2}{\delta j_3}r_1B_1 + \tfrac{r_2^2+j_1+\delta j_2}{\delta j_3}r_2B_3 = 0 \end{aligned} \tag{80}$$

Using Eqs. 58, 59, 70 and Eq. (74) to solving out for $G_1(\tau)$ and $G_2(\tau)$ we obtain

$$\begin{aligned} G_1(\tau) &= A_1\psi_1(\tau) + A_2\psi_2(\tau) \\ G_2(\tau) &= A_1\psi_3(\tau) + A_2\psi_4(\tau) \end{aligned} \tag{81}$$

where, $\Re_1$, $\Re_2$, $\Re_3$ and $\Re_4$ are as same as case 1. Furthermore,

$$\psi_1(\tau) = \Re_1\cosh(r_1\tau) + \Re_3\cosh(r_2\tau) + \theta_1$$

$$\psi_2(\tau) = \Re_2\cosh(r_1\tau) + \Re_4\cosh(r_2\tau) + \theta_2$$

$$\psi_3(\tau) = -\frac{r_1^2 + j_1 + \delta j_2}{\delta j_3}\Re_1\cosh(r_1\tau) - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3}\Re_3\cosh(r_2\tau) + \theta_3$$

$$\psi_4 = -\frac{r_1^2 + j_1 + \delta j_2}{\delta j_3}\Re_2\cosh(r_1\tau) - \frac{r_2^2 + j_1 + \delta j_2}{\delta j_3}\Re_4\cosh(r_2\tau) + \theta_4$$

The total normalized deflection at position η, and at time τ is:

$$\begin{aligned} V(\eta,\tau) &= A_1W_{\mathrm{I}}(\eta) + A_2W_{\mathrm{II}}(\eta) + G_1(\tau)W_{\mathrm{I}}(\eta) + G_2(\tau)W_{\mathrm{II}}(\eta) \\ &= A_1\varphi_1(\eta,\tau) + A_2\varphi_2(\eta,\tau) = A^T\varphi \end{aligned} \tag{82}$$

where

$$\varphi_1(\eta,\tau) = (1 + \psi_1(\tau))W_{\mathrm{I}}(\eta) + \psi_3(\tau)W_{\mathrm{II}}(\eta) \tag{83}$$

$$\varphi_2(\eta,\tau) = (1 + \psi_4(\tau))W_{\mathrm{II}}(\eta) + \psi_2(\tau)W_{\mathrm{I}}(\eta) \tag{84}$$

$$A^T = \begin{pmatrix} A_1 & A_2 \end{pmatrix}, \quad \varphi^T = \begin{pmatrix} \varphi_1(\eta,\tau) \\ \varphi_2(\eta,\tau) \end{pmatrix}$$

Hereinafter we let the A belong to the elli

$$Z(\theta,\varpi) = \left\{A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}^T : A^T\Omega A \leqslant \theta^2\right\} \tag{85}$$

where Ω is an 2×2 positive definite real symmetric matrix and θ^2 is a positive number. The shape and size of the ellipsoid are determined by the matrix Ω and value of θ^2, which are chosen to represent available information concerning the variability of the Fourier coefficients of the initial deflection profile. The set of extreme points of the set $Z(\theta, \Omega)$ is the ellipsoidal shell:

$$C(\theta,\Omega) = \left\{A = \begin{pmatrix} A_1 & A_2 \end{pmatrix}^T : A^T\Omega A = \theta^2\right\} \tag{86}$$

Let the Hamiltonian be:

$$H = A^T\varphi + \lambda(A^T\Omega A - \theta^2) \tag{87}$$

A necessary condition for an extremum is

$$\frac{\partial H}{\partial A} = \varphi + 2\lambda\Omega A = 0 \tag{88}$$

From Eqs. (78) and (80), we obtain λ as follows.

$$\lambda = \pm\frac{1}{2\theta}\sqrt{\varphi^T\Omega^{-1}\varphi} \tag{89}$$

Using Eqs. (82) and (87) we obtain the maximum value of the total normalized deflection $V_{\max}$ at the arbitrary position η of the two-span simply supported beam

$$V_{\max}(\xi,\tau) = \theta\sqrt{\varphi(\xi,\tau)^T\Omega^{-1}\varphi(\xi,\tau)} \tag{90}$$

5. Dynamic displacement responses

Figs. 1 and 2 depict variation of $w_{\max}$ as a function of η, when the value of the non-dimensional location $\alpha = a/L$ varies from 0.1 to 0.9 at time $\tau = 1$ with the increment of 0.1.

The curve is clearly symmetric with respect to 0.5. With the different support-placing, maximum displacement is different. It is clear that we can regulate the column displacement via different support-placing.

Figs. 1 and 2 depict the variation of the extremal values of the maximum displacement with the variation of a/L at specified time $\tau = 1$. We deduce that the minimum on this curve occurs at $a/L = 0.5$, whereas the maximum takes place at $a/L = 0.35$. Therefore, we conclude that the support should be placed at $a/L = 0.5$.

6. Variation of the displacement with time

In this section, we discuss the dynamic behavior of a two-span beam. Figs. 3 and 4 show the dynamic displacement over time, with the value for α to be set at 0.2.

From Fig. 3 we see that when $\delta = 0.75$, every point is periodically vibrating with time. The vibration curve of each point is close to the sinusoidal curve.

Fig. 4 shows the maximum total defection at $\eta = 0.3, 0.4, 0.5$ as a function of time. The axial load ratio is set at $\delta = 3$.

7. Extremal bending stress

The strength evaluation is usually one of the most important aspects in the structural analysis. Let us consider the bending stress

$$\sigma(\eta) = \frac{M(\eta)}{S} \tag{91}$$

where $S = I/h$ is the section modulus, h is the distance from the neutral surface of the farthest filer, from the neutral axis. For the bending moment we obtain

$$M(\eta) = A_1\Theta_1(\eta) + A_2\Theta_2(\eta) \tag{92}$$

$$\Theta_1(\eta) = \frac{d^2\varphi_1(\eta)}{d\eta^2}, \quad \Theta_2(\eta) = \frac{d^2\varphi_2(\eta)}{d\eta^2} \tag{93}$$

$$M(\eta) = \theta\sqrt{\Theta(\eta)^T\Omega^{-1}\Theta(\eta)} \tag{94}$$

Figs. 5 and 6 depict variation of $M(\eta)$ as a function of η, when the value of the non-dimensional location $\alpha = a/L$ varies from 0.1 to 0.9 at time instant set at $\tau = 1$ with the increment of 0.1.

The curves are clearly symmetric with respect of middle cross section. With the different support-placing, maximum displacement is different, as expected. It is clear that we can regulate the column stress via different support-placing.

Fig. 7 depicts the extremal value of the maximum stress ($\tau = 1$). When $\eta = 0.5$, the stress attains its minimum. As a result we can regulate the columns stress via supportplacing.

8. Evolution dynamic stress over time

Dynamic behavior of a two-span beam, is shown as following figures, the value chosen for $\alpha = 0.2$.

As Fig. 8 illustrates, when $\delta = 0.75$ that the stress curve is similar to the sinusoidal curve.

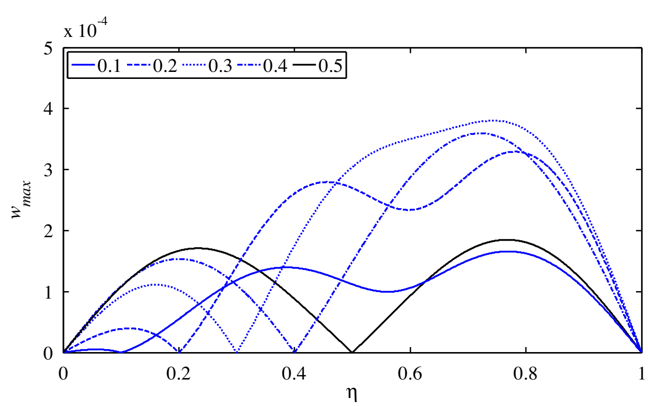

Fig. 1. Variation of w_{max} as a function of η ($\alpha = a/L = 0.1, 0.2, 0.3, 0.4, 0.5$, $\theta = 0.0001$, $\tau = 1$).

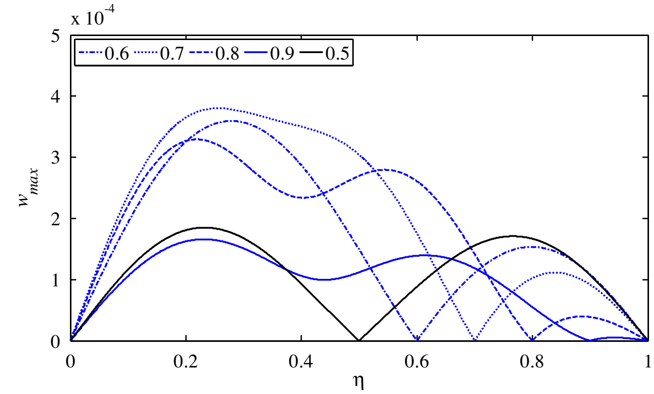

Fig. 2. Variation of w_{max} as a function of η ($\alpha = a/L = 0.5, 0.6, 0.7, 0.8, 0.9$, $\theta = 0.0001$, $\tau = 1$).

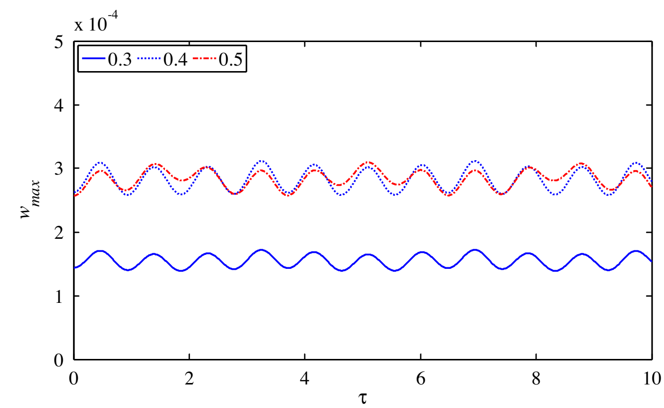

Fig. 3. Variation of the displacement vs. non-dimensional time coordinate ($\alpha = a/L = 0.2$; $\eta = 0.3, 0.4, 0.5$, $w_{max}(0.2) = 0$ $\delta = 0.75$).

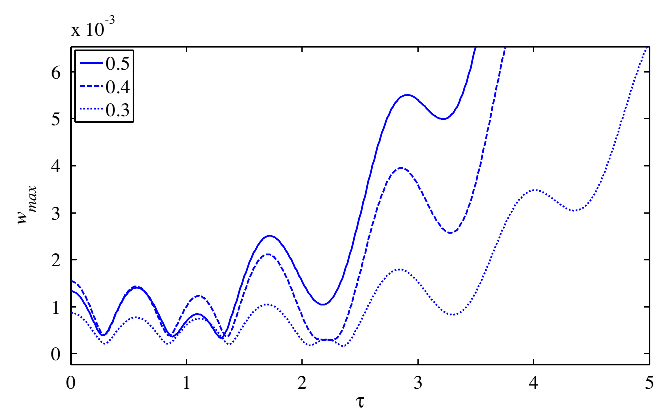

Fig. 4. Variation of the maximum total deflection with time ($\alpha = a/L = 0.2$; $\eta = 0.3, 0.4, 0.5$, $w_{max}(0.2) = 0$ $\delta = 3$).

Fig. 9 shows the stress at two different points on the beam ($\eta = 0.3$ and $\eta = 0.5$) as a function of time. The axial load ratio δ is fixed at three. Diagrams of this kind allow one to find the envelope of maximum stress as functions of both space and time.

9. Conclusion

In this study we present a systematic investigation of the problem of spacing an additional support so as to reduce either the

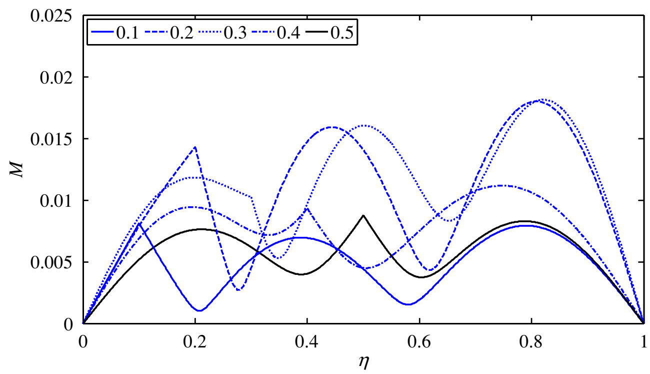

Fig. 5. Variation of $M(\eta)$ as a function of η ($\alpha = a/L = 0.1, 0.2, 0.3, 0.4, 0.5$, $\theta = 0.0001$, $\tau = 1$).

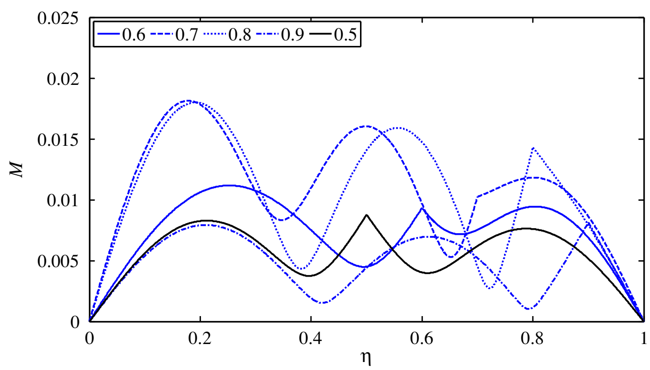

Fig. 6. Variation of $M(\eta)$ as a function of η ($\alpha = a/L = 0.5, 0.6, 0.7, 0.8, 0.9$, $\theta = 0.0001$, $\tau = 1$).

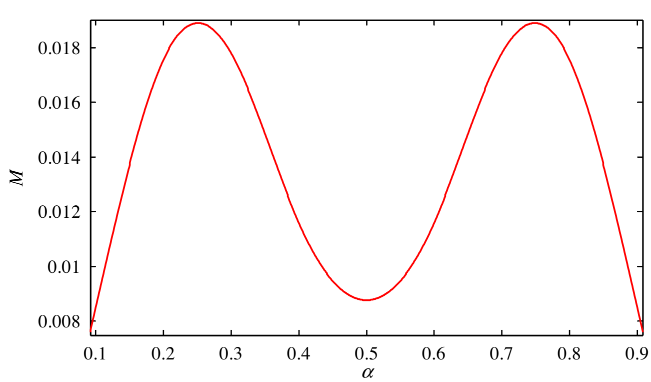

Fig. 7. The extremal value of the maximum stress vs. a/L ($\tau = 1$).

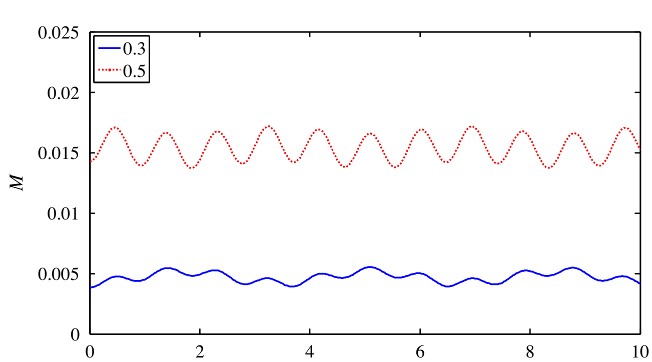

Fig. 8. Dynamic behavior of the two span Beam in time ($\alpha = a/L = 0.2$; $\eta = 0.3, 0.5$ $\delta = 0.75$).

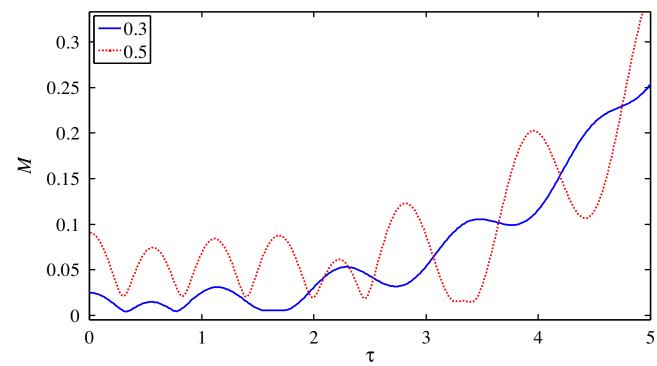

Fig. 9. Dynamic behavior of two-span beam in time ($\alpha = a/L = 0.2$; $\eta = 0.3, 0.5$ $\delta = 3$).

maximum displacement or maximum stress, apparently for the first time in the literature. The extreme responses of the structure are determined via the convex, non-probability analysis.

We conclude that one can regulate the dynamic behavior of a column with uncertain initial imperfections by the optimal placing of supports.

Acknowledgments

Xiaojun Wang thanks the National Natural Science Foundation of the PR China (No. 11002013), Defense Industrial Technology Development Program (No. A2120110001 and B2120110011) and 111 Project (No. B07009) for support. This work was initiated when Dr. J. Hu was a visiting Scholar at the Florida-Atlantic University, and when I. Elishakoff was serving as a Visiting Eminent Scholar of the BUAA. Warm and kind hospitality of International Division is recorded with thanks. Support of Chinese Scholarship Councel is gratefully appreciated.

References

Abovskii, N.P., Endzhievskii, L.V., Savchenkov, V.I., Deruga, A.P., Getz, I.I., 1993. Regulation. Optimization, Stroiizdat Publishers, Moscow (In Russian).
Ben-Haim, Y., Elishakoff, I., 1990. Convex Models of Uncertainty in Applied Mechanics. Elsevier, Amsterdam and New York.
Dekhtiar, A.S., 1991. Optimal support of plates on three points. Stroitelatvo i Arkhitektura 5, 108–111 (in Russian).
Elishakoff, I., 2005. Eigenvalues of Inhomogeneous Structures. CRC Press, Boca Raton.
Elishakoff, I., Ohsaki, M., 2010. Optimization and Anti-Optimization of Structure under Uncertainty. Imperial College Press, Singapore.
Katsikadelis, J.T., Tsiatas, G.C., 2006. Regulating the vibratory motion of beams using shape optimization. J. Sound Vib. 292, 390–401.
Åkesson, B., Olhoff, N., 1988. Minimum stiffness of optimally located supports for maximum value of beam eigen frequencies. J. Sound Vib. 120 (3), 457–463.
Liu, Z.S., Hu, H.C., Wang, D.J., 1994. Effect of small variation of support location on natural frequencies, 9–12, In: International Conference on Vibration Engineering, International Academic Publishers, Beijing.
Liu, Z.S., Hu, H.C., Huang, C., 2000. Derivative of buckling load with respect to support location. J. Eng. Mech. 126 (6), 559–564.
Neuringer, J., Elishakoff, I., 1998. Interesting instructional problems in column buckling for strength of materials and mechanics of solids courses. Int. J. Eng. Edu. 14 (3), 204–216.
Prager, W., Rozvany, G.I.N., 1975. Plastic design of beams: optimal locations of supports and steps in yield moment. Int. J. Mech. Sci. 17 (10), 627–631.
Qiu, Z.P., Wang, X.J., 2005. Two non-probabilistic set-theoretical models for dynamic response and buckling failure measures of bars with unknown-but-bounded initial imperfections. Int. J. Solids Struct. 42 (3–4), 1039–1054.
Rozvany, G.I.N., Mroz, Z., 1977. Column design: optimization of support conditions and segmentation. J. Struct. Mech. 5 (3), 279–290.
Szelag, D., Uróz, Z., 1978. Zeitschrift für angewandte Mathematik and Mekhanik 58 (11), 501–510.
Wang, C.M., Liev, K.M., Wang, L., Aug, K.K., 1992. Optimal locations of internal line supports for rectangular plates against buckling. Struct. Optim. 4, 199–205.
Won, K.M., Park, Y.S., 1998. Optimal support locations for structure to maximize its fundamental natural frequency. J. Sound Vib. 213, 810–812.